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Predicate Calculus with "undefined" as a Truth-Value

by

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We would like to use predicate calculus in the mathematical theory of computation. In particular, we would like to write formulas involving recursively defined predicates and functions. The trouble is that recursively defined predicates are not guaranteed to be defined for all values of their arguments, and therefore, it it not clear how to interpret formulas involving them.

We shall give an interpretation of predicate calculus formulas involving partial predicates and extend the notions of truth, valid formula and tautology. We have three truth values, t for <u>true</u>, f for <u>false</u>, and u for <u>undefined</u>. The well-formed formulas are the same as in predicate calculus except that we have a new propositional operator * defined by

$$*t = t$$
$$*f = t$$
$$*u = f$$

We shall call our system EPC.

The truth of a formula is determined from its constituents as follows:

1. An elementary form p(x,...,z) is <u>true</u>, <u>false</u> or <u>undefined</u> for each set of values of x,...,z.

2. The truth values of propositional combinations is determined by the truth-tables.

π	$\neg\pi$	$*\pi$
t	f	t
f	t	t
u	u	f

$\pi\rho$	$\pi\wedge\rho$	$\pi\vee\rho$	$\pi\supset\rho$	$\pi\equiv\rho$
tt	t	t	t	t
tf	f	t	f	f
tu	u	t	u	u
ft	f	t	t	f
ff	f	f	t	t
fu	f	u	t	u
ut	u	u	u	u
uf	u	u	u	u
uu	u	u	u	u

The truth values of the last two are in accordance with the definitions

$$\pi\supset\rho \approx \neg\pi\vee\rho$$

$$\pi\equiv\rho \approx (\pi\supset\rho)\wedge(\rho\supset\pi)$$

where $\approx$ is used in its ordinary sense as a meta mathematical symbol. They are the same as the conditional expression definitions of [1]. As explained in that paper the non-commutativity of $\pi\wedge\rho$ and $\pi\vee\rho$ arises from the convention that π is evaluated first, so that if π is false, ρ need not be evaluated to get $\pi\wedge\rho$.

3. $\forall x.\ \pi(x)$ is true if $\pi(x)$ is true for all x, undefined if $\pi(x)$ is undefined for some x and false otherwise.

4. $\exists x.\ \pi(x)$ is true if $\pi(x)$ is true for some x and is defined for all x, undefined if $\pi(x)$ is undefined for some x, and false otherwise.

If we consider formulas with no quantifiers we get an extended propositional calculus EPC. A formula is called a tautology if it is true for all values of its arguments. Ordinary tautologies are not tautologies of EPC since they are undefined if all the propositional variables are undefined. However, if π is an ordinary tautology, then $*\pi \supset \pi$ is a tautology of EPC so that formulas like

$$*(p \supset (q \supset p)) \supset (p \supset (q \supset p))$$

are tautologies. Whether a formula is a tautology can be determined by truth tables.

The equivalence of two formulas π and ρ is not expressed by $\pi \equiv \rho$ being a tautology, e.g. $p \equiv p$ is not a tautology. Therefore, we define

$$\pi \equiv \rho \leftrightarrow (*\pi \equiv *\rho) \wedge [*\pi \supset (\pi \equiv \rho)]$$

and $\pi \equiv \rho$ does express the equivalence of π and ρ. This is the strong equivalence of [1]. The weak equivalence of that paper is written

$$\pi \equiv_W \rho \leftrightarrow *\pi \wedge *\rho \supset . \pi \equiv \rho$$

where π and ρ do not involve *.

The following formulas are all tautologies

$$*(p \wedge q) \equiv *p \wedge (p \supset *q)$$
$$*(p \vee q) \equiv *p \wedge (\neg p \supset *q)$$
$$*(p \equiv q) \equiv *p \wedge *q$$
$$*\neg p \equiv *p$$
$$**p$$
$$*(p \equiv q)$$

The valid formulas of EFC in a domain are those which are true for all assignments of partial predicates to the predicate letters. Many questions about proof procedures for EFC are answered by a construction which gives for every formula π of EFC a formula π_1 of FC(the usual predicate calculus) such that π is valid in the domain if and only if π_1 is valid. Actually, we shall construct three formulas π_1, π_2 and π_3 of FC which are true when π is true, false, or undefined respectively. The construction is given by the following table.

π	π_1	π_2	π_3
$p(x,\ldots,z)$	$p_1(x,\ldots,z)$	$p_2(x,\ldots,z)$	$p_3(x,\ldots,z)$
$\neg\pi$	π_2	π_1	π_3
$*\pi$	$\pi_1 \vee \pi_2$	π_3	f
$\pi \wedge \rho$	$\pi_1 \wedge \rho_1$	$\pi_2 \vee (\pi_1 \wedge \rho_2)$	$\pi_3 \vee (\pi_1 \wedge \rho_3)$
$\pi \vee \rho$	$\pi_1 \vee (\pi_2 \wedge \rho_1)$	$\pi_2 \wedge \rho_2$	$\pi_3 \vee (\pi_2 \wedge \rho_3)$
$\pi \supset \rho$	$\pi_2 \vee (\pi_1 \wedge \rho_1)$	$\pi_1 \wedge \rho_2$	$\pi_3 \vee (\pi_1 \wedge \rho_3)$
$\pi \equiv \rho$	$(\pi_1 \wedge \rho_1) \vee (\pi_2 \wedge \rho_2)$	$(\pi_1 \wedge \rho_2) \vee (\pi_2 \wedge \rho_1)$	$\pi_3 \vee \rho_3$
$\forall x.\pi$	$\forall x.\pi_1$	$(\exists x.\pi_2) \wedge (\forall x.\neg\pi_3)$	$\exists x.\pi_3$
$\exists x.\pi$	$(\exists x.\pi_1) \wedge (\forall x.\neg\pi_3)$	$\forall x.\pi_2$	$\exists x.\pi_3$

That π_1, π_2 and π_3 have the required properties is obvious from the construction. This result shows that EFC is semi-decidable so that it should be possible to obtain a complete set of axioms and rules of inference. The method of semantic tableaux can be applied directly to EFC, but, at least in its most obvious form, it is impractical for all but the simplest examples because the number of cases that has to be considered increases rapidly.

The choice of definitions for the quantifiers requires some explanation. Our choice has the disadvantage that one cannot prove $\exists x.\pi(x)$ simply by exhibiting an a such that

$\pi(a)$ since if some $\pi(b)$ is undefined $\exists x.\pi(x)$ is considered undefined. However, the other possible quantifiers are definable in terms of ours. For example, we can define

$$(\epsilon x).\pi(x) \approx \exists x. \quad *\pi(x) \wedge \pi(x)$$

which has the above mentioned property.