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COMPUTING ESTIMATES FOR THE NUMBER OF BISECTIONS OF AN N x N CHECKERBOARD FOR N EVEN

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Abstract: This memo gives empirical justification for the assumption that the number of bisections of an N x N (N even) checkerboard is approximately given by the binomial coefficient $\binom{A}{\frac{A}{2}}$ where 2A is the length of the average bisecting cut.

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The Exhaustive Algorithm for Number of Bisections:

Problem: Find the number of distinctly different ways that a checkerboard may be divided (by continuously cutting along the edges of the interior squares) into two congruent halves. A bisection is distinctly different from another if it cannot be derived from the other by a rotation about or a reflection through the center of the board. We will limit n to be even.

Assuming that we are interested only in paths symmetric about the origin leads to the following algorithm for a total solution:

1. Number the interior vertices as follows:-

   a) The center is 0

   b) The vertex adjacent to 0 to its right is 1, that to its right is 2, etc., until $\frac{n}{2} - 1$

   c) The left most vertex of the row immediately above 0 is numbered $\frac{n}{2}$, that to its right $\frac{n}{2} + 1$

   d) This process is continued until the rest of the upper half of the board has been numbered

   e) The number of a lower half vertex is the negative of the number assigned to the vertex symmetrical to it through the center of the board.

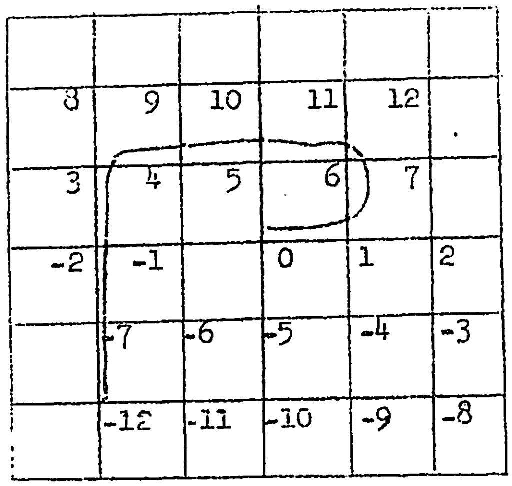
We now generate all possible half paths starting from 0-1 with the following restrictions:

a) A half-path must go up before it goes down

b) If a half-path uses vertex m the full path uses vertices m and -m

c) A half-path may use a vertex one time at most. It cannot use both vertices m and -m, where m is non-zero.

In the following: u is binary word which describes the use of the non-negative vertices; p is the current vertex; x is the x-coordinate of the current vertex. For example if a half path has proceeded as follows



p = -12

x = -2

u = 1000011111111

The n^{th} bit of u (starting from the right) is 1 if vertex n has been used by the full path. It is 0 otherwise.

Procedure Bisections (n) to count number of bisections

```
    REAL PROCEDURE C2(U,P,X); VALUE U,P,X; REAL P,X; BOOLEAN U;

            C2 ←

              (IF X = BIGX THEN 1

                ELSE IF REAL (U AND L([ABS(P+1)]) ≠ 0 THEN 0

                ELSE C2(U OR L[ABS(P+1)],P+1,X+1))  {Look Right}

            + (IF P > BIGP THEN 1

                ELSE IF REAL(U AND L[ABS(P+M)]) ≠ 0 THEN 0

                ELSE C2(U OR L[ABS(P+M)],P+M,X))   {Look Up}

            + (IF X = -BIGX THEN 1

                ELSE IF REAL(U AND L[ABS(P-1)]) ≠ 0 THEN 0

                ELSE C2(U OR L[ABS(P-1)],P-1,X-1))  {Look Left}

            + (IF P < -BIGP THEN 1

                ELSE IF REAL (U AND L[ABS(P-M)]) ≠ 0 THEN 0

                ELSE C2(U OR L[ABS(P-M)],P-M,X));   {Look Down}

     REAL PROCEDURE C1(U,X);  VALUE U,X; BOOLEAN U; REAL X;

            C1←

                C2(U OR L[X+M],X+M,X)+

              (IF X = BIGX THEN 1

                 ELSE C1(U OR L[X +1],X+1));
```

```
REAL PROCEDURE BISECTIØNS (N); VALUE N; REAL N;
 BEGIN
 BOOLEAN ARRAY L[0; N(N-2)/2];
 INTEGER SIZE, BIGX, BIGP, M;
 M←N - 1;
 BIGP←SIZE - M;
 SIZE←(N - 2) × N DIV 2;
 BIGX←N DIV 2 - 1;
 L[0] ← TRUE;
 FOR I← 0 STEP 1 UNTIL SIZE
 DO L[I+1]←BOOLEAN(2×REAL(L[I]));
 BISECTIØNS ←C1(BOOLEAN(3),1);
 END.
```

Application of the exhaustive algorithm to different boards yields the following results:-

n	number bisections
4	6
6	255
8	92263

Statistics of values obtained dynamically from the exhaustive algorithm are as follows with

L : the length of path

ℓ : the number of paths of length L.

E : the estimate of the total number of paths

$$E = \sum_{\text{paths} \in L} \frac{\prod_{p=1}^{m} (\text{number of vertices adjacent to vertex p unused by the full path})}{\ell}$$

σ : standard deviation of E

A : average path length

B : binomial coefficient $\binom{A}{A/2}$

V : true value for number of bisections

n	L	ℓ	E	σ	A	B	V
4					3.667	6	6
	2	1	1				
	3	2	<u>6</u>	0		E_m	
	4	1	12				
	5	2	24				
6					9.667	252	255
	3	1	2				
	4	3	16				
	5	9	46				
	6	10	99				
	7	21	255.43	99.79			E_m
	8	21	581				
	9	43	1343				
	10	38	2658				
	11	66	5588				
	12	22	6523				
	13	21	9590				

n	L	ℓ	E	σ	A	B	V
8					18.89	92378	92263
	4	1					
	5	4					
	6	16					
	7	45					
	8	72					
	9	166					
	10	239					
	11	542					
	12	768					
	13	1714	92548	75294	←——— E_m		
	14	2316					
	15	4947					
	16	5786					
	17	10972					
	18	9663					
	19	15563					
	20	10337					
	21	13822					
	22	6363					
	23	6603					
	24	1434					
	25	890					

One can see close similarity between B, E, V for the midlength path.

$$M = \frac{(n-2)n}{4} + 1$$

Each midlength path passes through 1/2 of the vertices of the checkerboard. The large value for σ is caused by the fact that E is calculated from sums of products of powers of 2, 3, and 1. These powers must sum to M but there is little else to restrict them. The following table shows the large variation in the individual factors.

n = 4

Factors: 6

n = 6

Factors: 486 324 216 144

n = 8

Factors: 354,294 236,196 118,098 157,464 104,976 78,732

69,984 52,488 46,656 34,992 31,104 26,244 23,328 20,736

17,496 15,552 13,824 13,122 11,664 10,368 7,776 6,912

Obtaining Estimates for 10 x 10 and 12 x 12 boards

On the assumption that there is a true correlation between B, V and E for the midlength path, I investigated the 10 x 10 and 12 x 12 boards.

In an attempt to investigate the paths in a random fashion I used pseudo random numbers to obtain at each vertex the order in which looks in the four directions were to be made. This was done by randomly choosing one of the permutations of the numbers 1, 2, 3, 4, e.g. if 4213 is chosen the order of looks is down, up, right, left. This took care of all vertices after the initial step upward. The rule I used for getting the initial step upward is that the probability a path goes up initially at vertex i is:

$$P[i] = \frac{2^{\frac{n-2}{2} - i}}{2^{\frac{n-2}{2}} - 1}$$

This is an unsophisticated guess given the empirical values:

n	P[1] ;	P[2] ;	P[3]
6	$\frac{170}{255}$	$\frac{85}{255}$	
8	$\frac{62362}{92263}$	$\frac{19937}{92263}$	$\frac{9964}{19937}$

n	Number Paths	Number Midpaths	Midpath Estimate	A
8		404	7.75×10^4	19
8		404	9.86×10^4	19
8		303	6.24×10^4	19
8		303	7.22×10^4	19
8		303	10.42×10^4	19
8		303	7.86×10^4	19
8		303	6.35×10^4	19
10		4400	2.60×10^8	
10		4400	1.81×10^8	
10		4400	2.41×10^8	
10	138122	300		31.3
10	14000	29		31.0
10	14000	48		31.2
10	28203	41		28.7
10	28000	160		30.9
12		1800	5.18×10^{12}	
12		4200	7.33×10^{12}	
12		4400	2.96×10^{12}	
12		8800	9.79×10^{12}	
12		8800	2.21×10^{12}	
12		9200	4.60×10^{12}	
12	60000	40		46.6
12	60957	200		44.9
12	60000	271		43.9
12	120000	1724		42.8

12	30000	41	46.8
12	30000	4	49.5
12	15000	4	48.6
12	15000	32	44.8
12	60000	28	44.6

Analysis of Estimates

Results for N = 8 show good consistency and indicate that the method may work for larger N.

Because the relative frequency of midpaths is low the computation of the average path length A was not computed concurrently with the midlength estimate. Determination of the latter was made comparatively efficient by terminating a subcalculation when the path length became greater than the midlength.

Results for N = 10 indicate a midlength estimate in the neighbourhood of 2×10^8 and an average pathlength of about 31. The value of B_{31} is 3×10^8.

Results for N = 12 show considerable variation in both the midlength estimate and the average path length. However there still appears to be correlation between the midlength estimate and the binomial estimate. The range of values for both is almost identical:

$$B_{44} = 2.1 \times 10^{12} \quad \text{and} \quad B_{48} = 1.3 \times 10^{13}$$

Consequently both estimates yield values in the range 5×10^{12}.

Conclusion

This investigation has been purely empirical, and there is no theoretical backup of the results. However, from the data obtained there does seem to be fair justification for the conclusion that the midlength estimate yields results remarkably similar to the binomial estimate involving the average path length. If either of these two can be shown theoretically to be good approximations to the number of bisections of the checkerboard, then the process used to investigate randomly paths yields comparatively stable and consistent values.