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DOCUMENTATION OF THE MACMAHON SQUARES PROBLEM

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Abstract: An exposition of the MacMahon Squares problem together with some "theoretical" results on the nature of its solutions and a short discussion of an ALGOL program which finds all solutions are contained herein.

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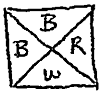
Part I Definitions and Theorems

(underlined words will be considered defined by context)

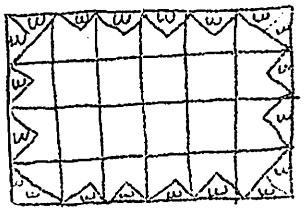
Def. 1: The problem: To fill a 4 x 6 matrix of positions with 24 colored squares according to the following rules:-

1) each square is quadresected by diagonals and each quadrant is colored with either red, white, or blue, e.g.



2) The twenty-four squares are all differently colored w.r.t. rotation but not reflexion.

3) The border of the matrix is made entirely of white quadrants, i.e.



4) When two squares are adjacent along an edge the colors of the corresponding quadrants must be the same.

Def. 2: Names:

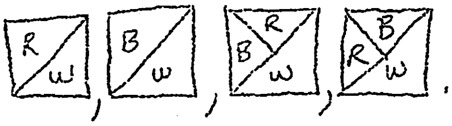
a) The white square is

(note: from henceforth if two adjacent quadrants of a square are the same color the portion of dividing diagonal will be omitted, e.g.)

b) The blue nick is

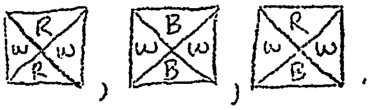
c) The red nick is

d) The diagonals are

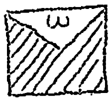


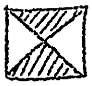
e) The above seven squares are called corner squares

f) The bowties are



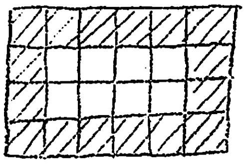
g) The stoppers are all pieces of the form:



(note: by convention shaded quadrants will represent red or blue and unlabeled quadrants will represent white, e.g.
the general bowtie is )

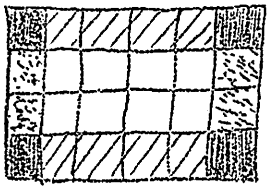
h) A colored square has no white quadrants and a W square has at least one white quadrent.

i) The border positions of the 4 x 6 matrix are those cross-hatched in the figure:



The center positions are the white positions.

j) The edge positions are cross-hatched in the following figure, the side positions are dotted and the corner positions are filled.



Other definitions will be introduced later.

Theorems

T1. There are only 24 possible different squares. Proof: trivial

T2. Counts.
  a) There are 18 W squares
  b) There are 7 Corner squares
  c) There are 8 Center positions

T3. 2 and only 2 W squares may occupy center positions

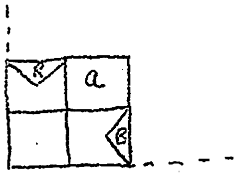
Proof: There are 18 W squares
- 16 must be used in border positions
2

T4. Only corner pieces may occupy corner positions. Proof: trivial

T5. The white square may not occupy a center position

Proof: If the white square is in a center position then it will require 2 W squares adjacent to it also in center positions. This makes 3 W squares in center positions. Contradiction.

T6.



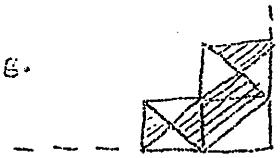
is impossible.

corner position in The matrix

Proof: A diagonal square must occupy position a. Thus the remaining 3 diagonals must go in the other corners. There can only be one more W square in the center positions. It cannot be a bow-tie since then yet another center W square would be required. Thus all three bowties must occupy border positions. This implies three more stoppers in the center. Contradictions.

Def. 3: A C-B (corner-bowtie) position is a diagonal in a corner with two bowties in the adjacent border positions.

e.g.



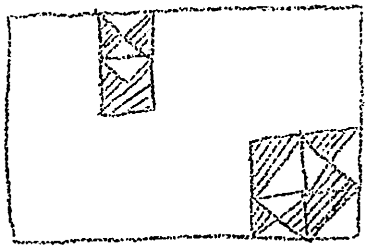
T7. Both of the two W squares in center positions cannot be stoppers.

Proof: Two stoppers in the center imply that all three bowties are in border positions. This leads to a Contradiction.

T8. No stoppers may occupy center positions.

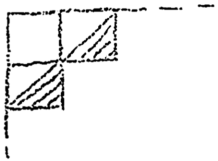
Proof: Two stoppers in center is impossible. Assume there is one stopper in a center position. This implies at least two bowties are in border positions. The third bowtie cannot occupy a center position since it would require a central W square adjacent to it, making three in the center.

Since three bowties are in border positions there are three quadrants facing in toward the center. Thus, in order for matchings to occur with only two central W's, two of the bowties must form a C-B position with a diagonal in the center adjacent position, and the third bowtie must be adjacent to the central stopper. Thus....



All the other corner squares besides the central diagonal must appear on the border and three of them must occupy the other three corners. Furthermore they cannot be placed on the border in such a fashion that any of their white quadrants are next to center positions. In particular the white square must occupy a corner, and adjacent to it must be two other corner squares. There are three possible cases

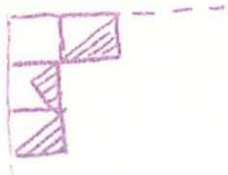
1.



with nicks in the other two corners.

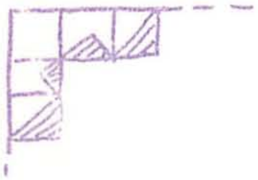
This is no good because there are not enough diagonals to be adjacent to the nicks.

2.



This is no good for there is only one more corner square to fill two more corner positions.

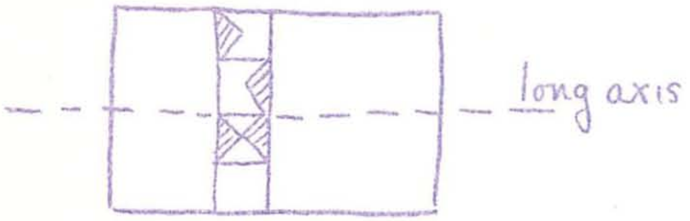
3.



This is no good for there are no more corner squares to fill two more corner positions.

Hence there can be no stopper in the center.

Def. 4: A set of four squares form a bridge if two are in edge positions which are mirror reflections along the long axis of the 4 x 6 matrix and each of which has a white quadrant next to its adjacent center position and the two squares which occupy these adjacent center positions have touching central white quadrants, e.g.

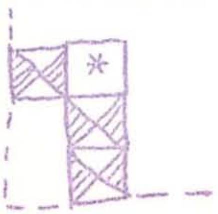


T9. If the two central W squares are bowties they are part of a bridge.

Proof: trivial.

T10. If exactly one bowtie occupies a center position it must be part of a bridge.

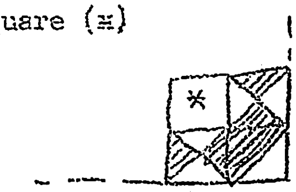
Proof: Assume no bridge. Then the two bowties on the border must be in C-B position. For if not then one must be adjacent to the corner bowtie. The other border bowtie must be in such a position that both of the white quadrants of it and the center bowtie must be adjacent to the same central corner piece (marked by an * in the diagram).



The * corner must be a diagonal for if it is a nick it will cause either a bridge to be formed or violate T3.

Similar reasoning as in T8 shows that the white square must be in a corner and a similar case analysis shows that there is no way to fill the other corners legally.

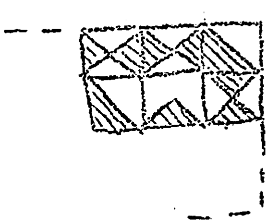
Hence the two border bowties must be in C-B position. If the corner square (*)



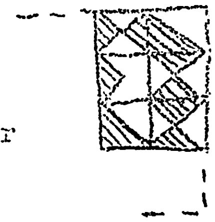
is a diagonal this will result in a violation of T3 since the central white quadrant of the center bowtie must be matched by a W. Hence the * must be a nick, which is furthermore adjacent to the center bowtie. The assumption that there is no bridge now violates T3.

T11. Every solution must contain a bridge

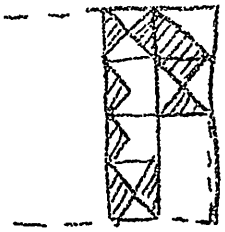
Proof: By contradiction assume there exists a solution with no bridge. This implies that all three bowties are on the border. T3 implies that two of them must be in C-B position, adjacent centrally to which must be some corner square. If this corner square is a diagonal, then the other bowtie must be adjacent to a central stopper which contradicts T8. Therefore this corner square must be a nick. Both the center unmatched white quadrant of the nick and the third bowtie must be adjacent to one center piece which by T3 and the assumption that there is no bridge must be a diagonal. Thus:



or



not



A similar argument to that in T8 shows that the white square must occupy a corner and a simple case analysis shows that there are not enough corner squares left to fill the other two corners legally.

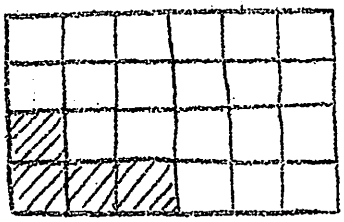
T12. There are no solutions with diagonals in all four corner positions.

Proof: The white square must appear on an edge flanked by the two nicks in such a fashion as to require the violation of T3.

T13. There are no solutions with the white square and both nicks occupying three of the four corner positions.

Proof: There are not enough diagonals.

T14. If the white square is always placed in one of the cross-hatched squares, there will be no duplication of solutions with respect to reflection or rotation.



Proof: trivial

T15. If the positions of the 4 x 6 matrix are ordered in some arbitrary fashion and solutions are constructed by placing squares into first positions on No.1, then position No.2 until position No.24, then duplication of solutions with respect to the interchanging of red and blue quadrants throughout will be eliminated by always placing the blue nick before the red nick and complying with the conditions of T14.

Proof: trivial.

Part II The program:

The program computes all possible solutions using ALGOL for the B5000 because of the ease of making recursive procedure definitions. The problem took about 40 computer hours to run to completion.

There are six possible ways of filling the corner positions:

type	0	white	nick	nick	diagonal
	1	white	nick	diagonal	diagonal
	2	white	diagonal	diagonal	diagonal
	3	nick	nick	diagonal	diagonal
	4	nick	diagonal	diagonal	diagonal
	5	diagonal	diagonal	diagonal	diagonal

Theorems 12 and 13 prove that there are no solutions of type 0 or 5; hence the program only considers types 1-4. For type 1 the first 200 solutions are printed out, and for the other three cases the first 50 solutions are printed out. In addition every time a new permutation for the corner positions is used the first solution of that case is also printed out. A total of 12261 solutions exists.

The program itself is basically very simple: the crux of it lies in the procedure GUTS.

GUTS uses the theorems proved above to limit possibilities so that it must examine only those cases which can lead to a solution.

GUTS(L) attempts to place a square into the Lth position. If it succeeds it removes that square from the list of available squares and calls GUTS(L+1). If it fails it tries successive available squares. If none will work it drops to GUTS(L-1) removing the L-1 th square previously placed from the available list since its placement leads to a deadend. When a solution is found i.e. when GUTS(24) goes to completion, control passes to a print routine and then to GUTS again to find the next solution. The order of filling positions is:-

4	12	11	10	9	3
13	17	19	21	23	16
14	18	20	22	24	15
1	5	6	7	8	2

The matrix drawn over solution No.1 type No.1 shows how the printout should be read. A transparent sheet with the lines inked would provide a convenient viewing roster. Naturally, a slight modification of the print routine would allow every solution to be printed out; however at the moment the necessary computer time is not available.

This is a copy of the computer output:

SOLUTION NUMBER 1